ISRAEL JOURNAL OF MATHEMATICS **148** (2005), 23–40

MULTIPLICATIVELY LARGE SETS AND ERGODIC RAMSEY THEORY

BY

VITALY BERGELSON*

Department of Mathematics, The Ohio State University
Columbus, OH 43210, USA
e-mail: vitaly@math.ohio-state.edu

Dedicated to MORI VAHAVERI Hillel Furstenberg
on the occasion of his retirement

ABSTRACT

Multiplicatively large sets are defined in $(\mathbb{N}, \cdot)$ by an analogy to sets of positive upper density in $(\mathbb{N}, +)$. By utilizing various ergodic multiple recurrence theorems, we show that multiplicatively large sets have a rich combinatorial structure. In particular, it is proved that for any multiplicatively large set $E \subset \mathbb{N}$ and any $k \in \mathbb{N}$, there exists $a, b, c, d, e, q \in \mathbb{N}$ such that

$$\{q^j(a+id) : 0 \le i, j \le k\} \subset E \quad \text{and} \quad \{b(c+ie)^j : 0 \le i, j \le k\} \subset E.$$

1. Introduction

For $r \in \mathbb{N}$, let $\mathbb{N} = \bigcup_{i=1}^r C_i$ be a partition of the positive integers. By van der Waerden's theorem ([W]), one of the C_i contains arbitrarily long arithmetic progressions. This result, one of Khintchine's "three pearls" of number theory ([K1]), has served as an impetus for numerous extensions and refinements. Perhaps the most famous among these generalizations is that due to Szemerédi: if $A \subseteq \mathbb{N}$ has positive upper density

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N} > 0,$$

* The author acknowledges support received from the National Science Foundation (USA) via grants DMS-0070566 and DMS-0345350.
Received January 9, 2004 and in revised form July 25, 2004

then A contains arbitrarily long arithmetic progressions ([S]).

In 1977 Hillel Furstenberg gave a completely different, ergodic-theoretical proof of Szemerédi's theorem ([F1]) and started a new mathematical area, Ergodic Ramsey Theory.

The ergodic approach has led to many new and powerful results, most of which still have no proof by conventional methods. (See, for example, [FK1], [FK2], [FK3], [BL1], [BL2], [L], [BM1], [BM3], [BL3], [FW1], [FW2].)

Both van der Waerden's and Szemerédi's theorems deal with the additive structure of the integers. In this paper we shall address, among other things, the following question: are there interesting results, similar to Szemerédi's theorem but pertaining to the multiplicative rather than additive structure on $\mathbb{N}$? It easily follows from van der Waerden's theorem that for any finite partition of $\mathbb{N}$, one of the cells of the partition contains arbitrarily long geometric progressions (just pick any $a \in \mathbb{N}$ and consider the restriction of the partition to the set $\{a^n : n \in \mathbb{N}\}$). On the other hand, the set of square-free numbers which, as is well known, has (additive) density $6/\pi^2$, obviously contains no three-term geometric progressions. So, in order to even formulate a multiplicative analogue of Szemerédi's theorem, one has to have at his disposal an appropriate notion of largeness, which is geared towards the multiplicative structure of the integers.

As we shall see in greater detail in the next section, each sequence of asymptotically invariant sets in $(\mathbb{N}, \cdot)$ (these are, roughly speaking, sets which do not change much under multiplication by integers, and will be defined more fully in Section 2) leads to a notion of multiplicative density which is good for our purposes. To concretize the discussion in this introduction, the reader may want to think of a set $A \subseteq \mathbb{N}$ as being multiplicatively large if for some sequence of positive integers $(a_n)_{n\in\mathbb{N}}$

$$\limsup_{n\to\infty} \frac{|A \cap a_n F_n|}{|a_n F_n|} > 0,$$

where $F_n = \{p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} : 0 \le i_j \le n, 1 \le j \le n\}$, and where the sequence $\{p_i\}$ consists of the primes in some arbitrarily preassigned order.

Remark 1.1: (i) In $(\mathbb{N}, +)$ one also has many notions of (additive) density, each corresponding to the averaging along a sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$. It is not hard to show that, for a set $A \subseteq \mathbb{N}$, Szemerédi's theorem implies (and is implied by) the fact that if, for a sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$, one has

$$\limsup_{n\to\infty} \frac{|A \cap [a_n, b_n]|}{b_n - a_n + 1} > 0,$$

then A contains arbitrarily long arithmetic progressions.

(ii) It is important to note that the notions of largeness based on additive and multiplicative densities do not overlap. For example, the set $2\mathbb{N}-1$ of odd natural numbers clearly has additive density $\frac{1}{2}$ along any sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$. On the other hand, it is not hard to show that this set has multiplicative density zero along any averaging scheme in $(\mathbb{N}, \cdot)$. Another example of this kind is provided by the set of square-free numbers mentioned above. In the other direction, consider the set $S = \bigcup_{n=1}^{\infty} a_n F_n$ where F_n are as defined above and the integers a_n satisfy $a_n > |F_n|$ for $n = 1, 2, \ldots$. It is easy to see that S has zero additive density with respect to any sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$. At the same time, S has multiplicative density 1 with respect to the sequence $(a_n F_n)_{n=1}^{\infty}$. (See Section 2 for more discussion and details.)

As may be expected by mere analogy, multiplicatively large sets can be shown to contain arbitrarily long geometric progressions. It will, however, be shown that multiplicatively large sets also contain various configurations which one expects to find in additively large sets, in particular arbitrarily long arithmetic progressions. As will become clear in Section 3, the somewhat unexpected presence of additive configurations in multiplicatively large sets is largely due to the fact that additive structures are preserved by dilations. On the other hand, since multiplicative configurations are not preserved by additive translations, an example of a result provable using our methods is that an additively large set contains additive translations of arbitrarily long geometric progressions.

We shall give now a sample of results which are obtained in this paper.

Definition 1.2: A finite set $S \subset \mathbb{N}$ is an **AG set of rank** k if there exist $q, a, d \in \mathbb{N}$ with $q > 1$ such that

$$S = \{q^j(a + id) : 0 \le i, j \le k\}.$$

Clearly, the set S above contains both arithmetic and geometric progressions. Hence the term **AG set**.

The following result is a special case of Theorem 3.11, which is proved in Section 3.

THEOREM 1.3: *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. Then E contains AG sets of arbitrarily large rank. In particular, E contains arbitrarily long arithmetic and geometric progressions.*

It follows from Theorem 1.3 that, for any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, any of the sets C_i which is multiplicatively large (and at least one C_i is such) contains

AG sets of arbitrarily large rank. While this partition result can also be obtained with the help of the two-dimensional version of the van der Waerden theorem, the proof of the following stronger statement utilizes more sophisticated multiple recurrence results. (See Theorem 3.12 below.)

THEOREM 1.4: *Let $r, n \in \mathbb{N}$. For any finite partition $\mathbb{N} = \bigcup_{s=1}^{r} C_s$, there exist $s \in \{1, 2, \ldots, r\}$, $a, b \in \mathbb{N}$ and $d, q \in C_s$ such that*

$$\{bq^j(a+id) : 0 \leq i, j \leq n\} \subset C_s.$$

In the following theorem (which appears as Theorem 3.15 in Section 3), arithmetic and geometric progressions are intertwined in a different fashion.

THEOREM 1.5: *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. For any $k \in \mathbb{N}$, there exist $a, b, d \in \mathbb{N}$ such that $\{b(a+id)^j, 0 \leq i, j \leq k\} \subset E$.*

The following two results involve IP sets. We remind the reader that, given an infinite set $S = \{n_i, i = 1, 2, \ldots\} \subseteq \mathbb{N}$, the additive IP set generated by S is defined as the set of all finite sums of elements of S with distinct indices:

$$IP^a(S) = FS(\{n_i\}_{i=1}^{\infty}) = \{n_{i_1} + n_{i_2} + \cdots + n_{i_k} : i_1 < i_2 < \cdots < i_k, k \in \mathbb{N}\}.$$

Similarly, the multiplicative IP set generated by S, $IP^m(S)$, is defined as the set of finite products:

$$IP^m(S) = FP(\{n_i\}_{i=1}^{\infty}) = \{n_{i_1} n_{i_2} \cdots n_{i_k} : i_1 < i_2 < \cdots < i_k, k \in \mathbb{N}\}.$$

Let $\mathcal{F}$ denote the set of finite non-empty subsets of $\mathbb{N}$. We shall find it convenient to index the elements of IP sets by elements of $\mathcal{F}$ and write the typical element of $IP^a(S)$ as $\sigma_\alpha := \sum_{i\in\alpha} n_i$ for $\alpha \in \mathcal{F}$, and the elements of $IP^m(S)$ as $\pi_\alpha := \prod_{i\in\alpha} n_i$ for $\alpha \in \mathcal{F}$. IP sets, as well as IP* sets, which will be defined in the next section, form a natural framework for various refinements of both Ramsey-theoretical and ergodic results. (See, for example, [FK2] and [BM3].) By invoking the IP polynomial Szemerédi theorem from [BM3], one can prove the following theorem. (See Theorem 3.9 below.)

THEOREM 1.6: *Let $B \subseteq \mathbb{N}$ be a multiplicatively large set and let*

$$F = \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}.$$

For any k polynomials $p_1(n), p_2(n), \ldots, p_k(n)$ which have positive leading coefficients and which satisfy $p_i(0) = 0$, $i = 1, 2, \ldots, k$, there exist $a, b \in \mathbb{N}$ such that $\{an_1^{p_1(b)}, an_2^{p_2(b)}, \ldots, an_k^{p_k(b)}\} \subset B$. Moreover, the set

$$\{b \in \mathbb{N} : \exists a \text{ with } \{an_1^{p_1(b)}, an_2^{p_2(b)}, \ldots, an_k^{p_k(b)}\} \subset B\}$$

is an additive IP set.*

Finally, we shall formulate a theorem which involves both additive and multiplicative IP sets. Note that it contains Theorem 1.3 as quite a special case. (Cf. Theorem 3.10 below.)

THEOREM 1.7: *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. For $k \in \mathbb{N}$, let $\sigma_\alpha^{(i)}$, $\alpha \in \mathcal{F}$, $i = 1, 2, \ldots, k$ be additive IP sets in $\mathbb{N}$, and let $\pi_\beta^{(j)}$, $\beta \in \mathcal{F}$, $j = 1, 2, \ldots, k$ be multiplicative IP sets. Then there exist $a, b \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{F}$ such that for all $i, j \in \{0, 1, \ldots, k\}$ one has $b\pi_\beta^{(j)}(a + \sigma_\alpha^{(i)}) \in E$.*

The structure of the paper is as follows. In the next section, we collect the definitions of various types of large sets and give formulations of ergodic facts which are needed for derivation of results of the type described in the introduction. Section 3 is devoted to proofs. As will become clear, most of the results are based on applying available multiple recurrence theorems such as those obtained in [FK2] and [BM3]. One of the author's goals in this paper was to achieve a high level of readability. Since some of the results discussed in Section 3 are special cases of others, it can be said that some results are stated multiple times. The author sincerely hopes that his attempt to make the paper accessible to as wide an audience as possible will not be perceived as an example of excessive verbosity.

ACKNOWLEDGEMENT: I would like to thank Hillel Furstenberg, Sasha Leibman and Ronnie Pavlov for their help in preparing this paper. I also thank the anonymous referee for constructive criticism and for suggesting an improvement in the formulation of Theorem 3.7.

2. Various types of large sets

In this section we shall summarize some facts about various notions of largeness which will be relevant for the discussion in the rest of the paper. Let $(G, \cdot)$ be a countably infinite (and for the purposes of this paper, always commutative and cancellative) semigroup. For a set $S \subseteq G$ and an element $g \in G$, let us define

$$S/g := \{x \in G : xg \in S\}.$$

A set $S \subseteq G$ is called **syndetic** if for some finite set $F \subset G$ one has $\bigcup_{g \in F} S/g = G$.

Syndetic sets in $(\mathbb{N}, +)$ or $(\mathbb{Z}, +)$ are just sets with bounded gaps, and frequently appear in various mathematical situations. (Here is a sample:

(i) H. Bohr's definition of almost periodicity, (ii) von Neumann's ergodic theorem, (iii) uniformly recurrent points in topological dynamical systems.) Syndetic sets in $(\mathbb{N},\cdot)$ are perhaps less intuitive. The goal of the following discussion is to help the reader to feel more comfortable with syndetic sets in $(\mathbb{N},\cdot)$.

We start with the trivial remark that if $S \subseteq (\mathbb{N},\cdot)$ is syndetic, then for any $m \in \mathbb{N}$ one has $S \cap m\mathbb{N} \neq \emptyset$. It immediately follows that, for a fixed $m \in \mathbb{N}$, none of the sets of the form $\{mn+i; n = 0,1,2,\ldots\}$, where $i = 1,2,\ldots,m-1$, is multiplicatively syndetic, while the set $m\mathbb{N}$ clearly is. This simple example shows also that an additive translation of a multiplicatively syndetic set may fail to be multiplicatively syndetic.

Our next example concerns the multiplicative analogue of the partition of $\mathbb{N}$ into classes of residues modulo m. Let

$$S_i = \{p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} : \sum_{j=1}^{k} s_j = i \bmod m\}, \quad i = 0,1,\ldots,m-1.$$

Clearly, $\mathbb{N} = \bigcup_{i=0}^{m-1} S_i$ and each S_i is multiplicatively syndetic.

Finally, consider the set $A = \bigcup_{k=1}^{\infty} [2^{2k}, 2^{2k+1})$, where the intervals are taken in $\mathbb{N}$. Since for any $n \in \mathbb{N}$ either n or $2n$ belongs to A, A is multiplicatively syndetic. It is also obvious that A is not additively syndetic.

One usually meets syndetic sets either in the hypotheses of theorems or in their conclusions. When syndetic sets occur in the hypotheses, one would like to replace syndeticity by a weaker property, which often turns out to be the property of having positive density. For example, Szemerédi's theorem extends the van der Waerden theorem to the sets of positive density in $(\mathbb{N},+)$. (Note that the van der Waerden theorem, in one of its equivalent forms, states that any syndetic set in $(\mathbb{N},+)$ contains arbitrarily long arithmetic progressions.)

On the other hand, when syndetic sets appear as a part of the conclusion, one would like to replace syndeticity with a stronger notion of largeness which, desirably, should have the finite intersection property. For example, the so-called Khintchine's recurrence theorem (see [K2], and [B2], Section 5) states that, for any probability measure preserving system $(X,\mathcal{B},\mu,T)$ and for any $\epsilon > 0$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, the set

$$R_A = \{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > (\mu(A))^2 - \epsilon\}$$

is syndetic in $(\mathbb{N},+)$. One can, however, show that the set R_A has many additional features and, in particular, the collection of sets $(R_A)_{A\in\mathcal{B}}$ has the finite intersection property. (See, for example, [B2], pp. 35–36 and 49–50.)

The notion of an IP* set, which we shall presently introduce, is one of these desirable strengthenings of syndeticity and has already proved useful in ergodic theory and combinatorics. (See, for example, [F2] ch. 9, [FK2], [BM3].) In the following definition, we utilize the notation introduced in Section 1.

Definition 2.1: A set $A \subseteq \mathbb{N}$ is an additive (multiplicative) IP* set if, for any infinite set $S \subseteq \mathbb{N}$, one has $IP^a(S) \cap A \neq \emptyset$ (correspondingly, $IP^m(S) \cap A \neq \emptyset$).

By invoking Hindman's theorem ([H]), which states that, for any finite partition of $\mathbb{N}$, one of the cells of the partition has to contain an IP set, one can show that, if A is an additive (multiplicative) IP* set, then for any infinite $S \subseteq \mathbb{N}$, A intersects with $IP^a(S)$ ($IP^m(S)$) along an IP set. This, in turn, implies that IP* sets have the finite intersection property. It is also not hard to see that IP* sets are syndetic. On the other hand, it is easy to see that not every syndetic set is IP^*. (Consider, for example, the set $2\mathbb{N} - 1$ in $(\mathbb{N}, +)$ and the set $2\mathbb{N}$ in $(\mathbb{N}, \cdot)$.)

Remark 2.2: The real key to both IP and IP* sets is provided by the topological algebra in $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$. See [B2], [B4], and [HiS] for more information.

The notions of syndeticity and IP*-ness make sense in any semigroup. We shall discuss now a different notion of size which can be introduced only for *amenable* semigroups.

Definition 2.3: Let $(G, \cdot)$ be a discrete semigroup. Denote by $\mathcal{P}(G)$ the set of all subsets of G. The semigroup G is called **amenable** if there exists a finitely additive probability measure μ on $\mathcal{P}(G)$ satisfying $\mu(A) = \mu(A/x)$ for all $A \in \mathcal{P}(G)$ and $x \in G$.

It is well known that all abelian semigroups are amenable and that G is amenable if and only if there exists an invariant mean for G, namely a positive linear functional L on the space $B(G)$ of bounded real-valued functions on G satisfying the following conditions:

(i) $L(1_G) = 1$.

(ii) $L(f_s) = L(f)$ for all f in $B(G)$ and s in G, where $f_s(x) := f(xs)$.

If G is an abelian countable semigroup with cancellation law (such as $(\mathbb{N}, +)$ or $(\mathbb{N}, \cdot)$), amenability is also equivalent to the following property:

(iii) G possesses a Følner sequence, i.e. a sequence of finite sets $F_n \subset G$ such that for any $g \in G$ one has

$$\frac{|gF_n \triangle F_n|}{|F_n|} = \frac{|F_n \triangle (F_n/g)|}{|F_n|} \underset{i\to\infty}{\longrightarrow} 0.$$

For example, in $(\mathbb{N}, +)$, any sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$ is a Følner sequence.

As for $(\mathbb{N}, \cdot)$, let $(a_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathbb{N}$ and let

$$F_n = \{a_n p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} : 0 \le i_j \le k_{j,n}, j = 1, 2, \ldots, n\},$$

where $k_{j,n}$ is a doubly indexed sequence of positive integers such that, for every j, $k_{j,n} \to \infty$ as $n \to \infty$, and $\{p_n\}$ is the set of primes.

It is not hard to check that the sets F_n form a Følner sequence in $(\mathbb{N}, \cdot)$. Given a Følner sequence $\{F_n\}$ and a set A in G, one defines the (upper) density of A with respect to $\{F_n\}$ as

$$\overline{d}_{\{F_n\}}(A) = \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

One can show that a set $A \subseteq G$ has positive upper density with respect to some Følner sequence if and only if there exists an invariant mean L on $B(G)$ such that $L(1_A) > 0$. The following two immediate properties of density $\overline{d}_{\{F_n\}}$ will be frequently used in the sequel.

(i) If $\overline{d}_{\{F_n\}}(A) > 0$ and $A = A_1 \cup A_2 \cup \cdots \cup A_r$, then one of the A_i satisfies $\overline{d}_{\{F_n\}}(A_i) > 0$.

(ii) For any Følner sequence $\{F_n\}$ for $(\mathbb{N}, \cdot)$, $A \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, one has

$$\overline{d}_{\{F_n\}}(A) = \overline{d}_{\{F_n\}}(mA) = \overline{d}_{\{F_n\}}(A/m).$$

We shall call sets which have positive density with respect to a Følner sequence in $(\mathbb{N}, \cdot)$ (respectively, $(\mathbb{N}, +)$) **multiplicatively** (respectively, **additively**) **large**.

In his proof of Szemerédi's theorem ([F1]), Furstenberg established a connection between additively large sets and measure preserving systems. One can verify that Furstenberg's correspondence principle actually applies to any discrete amenable semigroup (cf., for example, [BM2], Theorem 2.1 and [B3], Theorem 6.4.17.) The following version of Furstenberg's correspondence principle for large sets in $(\mathbb{N}, \cdot)$ will be utilized in the next section.

THEOREM 2.4: *Let $\{F_n\}$ be a Følner sequence in $(\mathbb{N},\cdot)$ and assume that $E \subseteq \mathbb{N}$ is such that $\overline{d}_{\{F_n\}}(E) > 0$. Then there exists a probability space $(X,\mathcal{B},\mu)$, a measure preserving $(\mathbb{N},\cdot)$-action $(T_n)_{n\in\mathbb{N}}$ on X and a set $A \in \mathcal{B}$ with $\mu(A) = \overline{d}_{\{F_n\}}(E)$ such that for any $k \in \mathbb{N}$ and any $n_1, n_2, \ldots, n_k \in \mathbb{N}$ one has*

$$\overline{d}_{\{F_n\}}(E/n_1 \cap E/n_2 \cap \cdots \cap E/n_k) \geq \mu(T_{n_1}^{-1}A \cap T_{n_2}^{-1}A \cap \cdots \cap T_{n_k}^{-1}A).$$

In particular, for any $n_1, n_2, \ldots, n_k$ such that $\mu(T_{n_1}^{-1}A \cap T_{n_2}^{-1}A \cap \cdots \cap T_{n_k}^{-1}A) > 0$, one has $E/n_1 \cap E/n_2 \cap \cdots \cap E/n_k \neq \emptyset$ and hence for some $m \in \mathbb{N}$, $E \supset \{mn_1, mn_2, \ldots, mn_k\}$.

It follows from the Furstenberg–Katznelson IP Szemerédi theorem, proved in [FK2], that, for any $k \in \mathbb{N}$, any k commuting measure preserving actions $(T_g^{(i)})_{g\in G}$, $i = 1, 2, \ldots, k$, of an abelian group G on a probability space $(X,\mathcal{B},\mu)$, and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $c > 0$ such that the set

$$\{g \in G : \mu(A \cap T_g^{(1)}A \cap T_g^{(2)}A \cap \cdots \cap T_g^{(k)}A) > c\}$$

is an IP* set in G. (IP and IP* sets in any abelian semigroup are defined in complete analogy to IP and IP* sets in $\mathbb{N}$.)

It is not hard to verify that this result extends to measure preserving actions of cancellative abelian semigroups. In the next section, it will be utilized for measure preserving $(\mathbb{N},+)$- and $(\mathbb{N},\cdot)$-actions and often referred to as the IP Szemerédi theorem (although the main theorem proved in [FK2] is much stronger).

We shall also use in Section 3 the following combinatorial result, which follows from the IP Szemerédi theorem via Furstenberg's correspondence principle.

THEOREM 2.5: *Let E_1, E_2 be, respectively, additively and multiplicatively large sets in $\mathbb{N}$, and let $k \in \mathbb{N}$. Then:*

(i) *The set of differences of length k arithmetic progressions contained in E_1 is an additive IP* set.*

(ii) *The set of ratios of length k geometric progressions contained in E_2 is a multiplicative IP* set.*

The following stronger combinatorial result also follows from [FK2] and will be needed for the proof of Theorem 1.7 formulated in the Introduction.

THEOREM 2.6: *Let E_1, E_2 be, respectively, additively and multiplicatively large sets in $\mathbb{N}$, and let $k \in \mathbb{N}$. Then:*

(i) *For any additive IP sets $(\sigma_\alpha^{(i)})_{\alpha\in\mathcal{F}}$, $i = 1, 2, \ldots, k$, there exist $a \in E_1$ and $\alpha \in \mathcal{F}$ such that*

$$\{a, a + \sigma_\alpha^{(1)}, a + \sigma_\alpha^{(2)}, \ldots, a + \sigma_\alpha^{(k)}\} \subset E_1.$$

(ii) *For any multiplicative IP sets* $(\pi_\alpha^{(i)})_{\alpha\in\mathcal{F}}$, $i = 1, 2, \ldots, k$, *there exist* $b \in E_2$ *and* $\beta \in \mathcal{F}$ *such that*

$$\{b, b\pi_\beta^{(1)}, b\pi_\beta^{(2)}, \ldots, b\pi_\beta^{(k)}\} \subset E_2.$$

Finally, we formulate a corollary from the IP polynomial Szemerédi theorem ([BM3]), which we will also use in the next section.

THEOREM 2.7: *Let* $k \in \mathbb{N}$ *and let* $T_1, T_2, \ldots, T_k$ *be commuting measure preserving transformations acting on a probability space* $(X, \mathcal{B}, \mu)$. *For any* $A \in \mathcal{B}$ *with* $\mu(A) > 0$, *and any polynomials* $p_1(n), p_2(n), \ldots, p_k(n) \in \mathbb{Z}[n]$ *with zero constant terms, the set*

$$\{n : \mu(A \cap T_1^{-p_1(n)} A \cap T_2^{-p_2(n)} A \cap \cdots \cap T_k^{-p_k(n)} A) > 0\}$$

is an additive IP set. (If the transformations* T_i *are not invertible, then it is assumed that the polynomials* $p_i(n)$ *have positive leading coefficients.)*

3. Combinatorial richness of multiplicatively large sets

The following result, obtained in [B1], will be repeatedly used in this section.

THEOREM 3.1: *Let* $(X, \mathcal{B}, \mu)$ *be a probability space and let, for some* $c > 0$, $A_n \in \mathcal{B}$, $n = 1, 2, \ldots$, *be sets satisfying* $\mu(A_n) \geq c$ *for all* n. *Then there exists a set* $S \subseteq \mathbb{N}$ *with*

$$\bar{d}(S) = \limsup_{N\to\infty} \frac{|S \cap \{1, 2, \ldots, N\}|}{N} \geq c$$

such that for any finite subset $F \subset S$ *one has* $\mu(\bigcap_{n\in F} A_n) > 0$.

Note that, by Szemerédi's theorem, the set S featured in the formulation of Theorem 3.1 contains arbitrarily long arithmetic progressions. This leads to the following immediate application.

THEOREM 3.2: *Any multiplicatively large set* $E \subseteq \mathbb{N}$ *contains arbitrarily long arithmetic progressions.*

Proof: Invoking Furstenberg's correspondence principle, as formulated in Theorem 2.4 above, let $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{N}})$ be the corresponding measure preserving system and let $A \in \mathcal{B}$ be the set of positive measure corresponding to E. Let $A_n = T_n^{-1}A$. Clearly, $\mu(A_n) = \mu(A)$ for all $n \in \mathbb{N}$. By Theorem 3.1 there exists an additively large set S with the property that for any finite $F \subset S$ one has $\mu(\bigcap_{n\in F} T_n^{-1}A) > 0$. Using Szemerédi's theorem, we get, for arbitrary

$k \in \mathbb{N}$, an arithmetic progression $P_k = \{n + id, i = 0, 1, \ldots, k-1\} \subset S$ such that

$$\mu\Big(\bigcap_{n \in P_k} T_n^{-1} A\Big) > 0.$$

Applying again Furstenberg's correspondence principle, we see that the set $\bigcap_{n \in P_k} E/n$ is multiplicatively large and, in particular, non-empty. This implies that for some $m \in \mathbb{N}$, $E \supset mP_k$. ∎

Remark 3.3: Note that it follows from Furstenberg's correspondence principle that for any finite set $F \subset S$ the set $\bigcap_{n \in F} E/n$ is multiplicatively large. This fact will be utilized in the proof of Theorem 3.10 below.

By utilizing in the proof of Theorem 3.2 the IP version of Szemerédi's theorem (see Theorem 2.5(i)), one gets the fact that, for a fixed $k \in \mathbb{N}$ and IP set $IP^a(S)$, any multiplicatively large set contains progressions of the form $\{m(a+id), i = 1, 2, \ldots, k-1\}$ where $d \in IP^a(S)$. This fact, in its turn, is contained in the following stronger statement, the proof of which incorporates Theorem 2.7.

THEOREM 3.4: *Let E be a multiplicatively large set. For any $k \in \mathbb{N}$, additive IP set $IP^a(S)$, and polynomials $p_1(n), p_2(n), \ldots, p_k(n) \in \mathbb{Z}[n]$ satisfying $p_i(0) = 0$, $i = 1, 2, \ldots, k$, there exist $a, b \in \mathbb{N}$ and $d \in IP^a(S)$ such that $E \supset \{b(a + p_i(d)), i = 1, 2, \ldots, k\}$.*

As a matter of fact, Theorems 3.2 and 3.4 are special cases of the following general proposition, which can be proved by the same method.

THEOREM 3.5: *Let $S \subset \mathcal{F}$ be a family of finite sets with the property that any additively large set in $\mathbb{N}$ contains a configuration of the form $a + F$, where $F \in S$. Then any multiplicatively large set contains a configuration of the form $b(a + F)$, where $a, b \in \mathbb{N}$, $F \in S$.*

Following I. Ruzsa, let us call a set $S \subseteq \mathbb{N}$ **intersective** if for any additively large set $E \subseteq \mathbb{N}$ one has $(E - E) \cap S \neq \emptyset$. (Here, $E - E$ is the set of positive differences of members of E.) It is not too hard to check that any set containing either arbitrarily long progressions of the form $\{d, 2d, \ldots, kd\}$ or, more generally, arbitrarily large sets of sums $\{\sum_{i=1}^{k} \epsilon_i b_i, \epsilon_i = 0, 1, i = 1, 2, \ldots, k\}$ is an intersective set. It follows from Theorem 3.2 that for any additively large set $E \subseteq \mathbb{N}$ and any multiplicatively large set $B \subseteq \mathbb{N}$ one has $(E - E) \cap (B - B) \neq \emptyset$. This fact is a special case of the following stronger proposition.

THEOREM 3.6: *For a set $C \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, denote by $D_k(C)$ the set of differences of length k arithmetic progressions contained in C:*

$$D_k(C) = \{d \in \mathbb{N} : \exists c \in C : c, c+d, \dots, c+(k-1)d \in C\}.$$

Let $n, k \in \mathbb{N}$ be fixed. For any additively large sets $E_1, E_2, \dots, E_n \subseteq \mathbb{N}$ and any multiplicatively large set $B \subseteq \mathbb{N}$ one has

$$D_k(E_1) \cap D_k(E_2) \cap \cdots \cap D_k(E_n) \cap D_k(B) \neq \emptyset.$$

Proof: To make the proof shorter and the main ideas more transparent, we give a proof using ergodic language. We remark first that if a set $S \subseteq \mathbb{N}$ contains arbitrarily long arithmetic progressions of the form $\{t, 2t, \dots, kt\}$, then for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $d \in S$ such that

$$\mu(A \cap T^{-d}A \cap T^{-2d}A \cap \cdots \cap T^{-kd}A) > 0.$$

(This actually follows from Szemerédi's theorem. See, for example, [BHMP], Theorem F2, p. 548.) An example of a set S with this property is $S = D_k(C)$ for C such that, for *every* k, $D_k(C) \neq \emptyset$. Let now $(X_i, \mathcal{B}_i, \mu_i, T_i)$ and $A_i \in \mathcal{B}$ with $\mu_i(A_i) > 0$ be the measure preserving systems and sets of positive measure which correspond to the sets E_i via Furstenberg's correspondence principle (applied here to the additively large sets E_i.) Let $(X, \mathcal{B}, \mu, T)$ be the product system and let $A = A_1 \times A_2 \times \cdots \times A_n$. By Theorem 3.2 and the remark above,

$$\{d \in \mathbb{N} : \mu(A \cap T^{-d}A \cap T^{-2d}A \cap \cdots \cap T^{-kd}A) > 0\} \cap D_k(B) \neq \emptyset,$$

which implies that

$$\bigcap_{i=1}^{n} \{d \in \mathbb{N} : \mu_i(A_i \cap T_i^{-d}A_i \cap T_i^{-2d}A_i \cap \cdots \cap T_i^{-kd}A_i) > 0\} \cap D_k(B) \neq \emptyset.$$

Returning back to the sets E_i, we get by the Furstenberg correspondence principle that $D_k(E_1) \cap D_k(E_2) \cap \cdots \cap D_k(E_n) \cap D_k(B) \neq \emptyset$. ∎

The following theorem may be viewed as a sort of weak dual form of Theorem 3.6.

THEOREM 3.7: *For a set $C \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, denote by $R_k(C)$ the set of ratios of length k geometric progressions contained in C:*

$$R_k = \{q : \exists c \in C : c, cq, cq^2, \dots, cq^{k-1} \in C\}.$$

For any multiplicatively large sets $B_1, B_2, \ldots, B_n \subseteq \mathbb{N}$, any additively large set E, and any $l \in \mathbb{N}$, one has $R_k(B_1) \cap R_k(B_2) \cap \cdots \cap R_k(B_n) \cap D_l(E) \neq \emptyset$.

Proof: We are going to imitate the proof of Theorem 3.6. We remark first that for any additively large set E and $k, l \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\{n, n^2, \ldots, n^k\} \subset D_l(E)$. This fact follows immediately from the polynomial Szemerédi theorem (see [BL1], [BM1]), which implies that, for any finite set of polynomials $p_i(n) \in \mathbb{Z}[n]$, $i = 1, 2, \ldots, m$, with $p_i(0) = 0$ for each i, there exists a such that $\{a, a + p_1(n), a + p_2(n), \ldots, a + p_m(n)\} \subset E$.

Our next remark is that if a set $S \subseteq \mathbb{N}$ contains arbitrarily long geometric progressions of the form $\{m, m^2, \ldots, m^k\}$, then for any measure preserving action of $(\mathbb{N}, \cdot)$ on a probability space $(X, \mathcal{B}, \mu)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in S$ such that $\mu(A \cap T_n^{-1} A \cap T_{n^2}^{-1} A \cap \cdots \cap T_{n^k}^{-1} A) > 0$. (The existence of $n \in \mathbb{N}$ with $\mu(A \cap T_n^{-1} A \cap T_{n^2}^{-1} A \cap \cdots \cap T_{n^k}^{-1} A) > 0$ follows, for example, from the corollary of the Furstenberg–Katznelson IP Szemerédi theorem formulated at the end of Section 2. The fact that n can be chosen from S follows from an argument analogous to that used in the proof of Theorem F2 in [BHMP].)

The rest of the proof is similar to that of Theorem 3.6 and is left to the reader. ∎

The results obtained so far in this section pertain mostly to configurations of an additive nature found in multiplicatively large sets. Direct application of available multiple recurrence theorems (such as those obtained in [FK2], [BL1], or [BM3]) shows that multiplicatively large sets contain plentiful multiplicative configurations as well. One of the fruitful approaches to the study of configurations in large sets is to look for "geometric" images of finite sets. For example, it is not hard to verify that Szemerédi's theorem is equivalent to the fact that if $E \subseteq \mathbb{N}$ is an additively large set, then for any finite set $F \subset \mathbb{N}$ one can find an affine image of F inside E. In other words, for any finite set F there are $a, b \in \mathbb{N}$ such that $a + bF = \{a + bn, n \in F\} \subset E$.

In the following proposition, we collect some similar facts about multiplicatively large sets.

THEOREM 3.8: *Let $B \subseteq \mathbb{N}$ be a multiplicatively large set and let*

$$F = \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}.$$

Then:

(i) *There exist $a, b \in \mathbb{N}$ such that $a + bF = \{a + bn, n \in F\} \subset B$.*

(ii) *There exist $a, b \in \mathbb{N}$ such that $\{an^b, n \in F\} \subset B$. Moreover, the set $\{b \in \mathbb{N} : \exists a$ with $an^b \in B, n \in F\}$ is an additive IP* set.*

(iii) *There exist $a, b \in \mathbb{N}$ such that $\{ab^n, n \in F\} \subset B$. Moreover, the set $\{b \in \mathbb{N} : \exists a$ with $ab^n \in B, n \in F\}$ is a multiplicative IP* set.*

Remark: It clearly suffices to consider F of the form $\{1, 2, \ldots, k\}$.

Proof: Statement (i) follows immediately from Theorem 3.2. As for statements (ii) and (iii), they both follow — via the Furstenberg correspondence principle — from an appropriately chosen version of the IP Szemerédi theorem. Accordingly, we shall confine our discussion to the relevant ergodic statements. Let $(T_n)_{n\in\mathbb{N}}$ be a measure preserving action of $(\mathbb{N}, \cdot)$ on a probability space $(X, \mathcal{B}, \mu)$. Let $T_{n_1}, T_{n_2}, \ldots, T_{n_k}$ be the elements of the action $(T_n)_{n\in\mathbb{N}}$ corresponding to the elements of F. By the IP Szemerédi theorem, applied to commuting $(\mathbb{N}, +)$-actions $(T^b_{n_i})_{b\in\mathbb{N}}$, we see that for any $A \in \mathcal{B}$ with $\mu(A) > 0$ the set

$$\{b \in \mathbb{N} : \mu(A \cap T^{-b}_{n_1} A \cap T^{-b}_{n_2} A \cap \cdots \cap T^{-b}_{n_k} A) > 0\}$$

is an additive IP* set. To see that this gives us (ii), it remains only to rewrite this set as $\{b \in \mathbb{N} : \mu(\bigcap_{n\in F} T^{-1}_{n^b} A) > 0\}$ and to apply the Furstenberg correspondence principle.

Similarly, statement (iii) follows by considering commuting $(\mathbb{N}, \cdot)$-actions $(T_{b^{n_i}})_{b\in\mathbb{N}}$, $i = 1, 2, \ldots, k$, where, as before, n_i are the elements of F. Applying again the IP Szemerédi theorem, we obtain the fact that the set $\{b : \mu(\bigcap_{n\in F} T^{-1}_{b^n} A) > 0\}$ is a multiplicative IP* set, which gives (iii). ∎

By using the IP polynomial Szemerédi theorem ([BM3]), one can obtain a further refinement of Theorem 3.8. Here is, for example, a polynomial extension of statement (ii).

THEOREM 3.9: *Let $B \subseteq \mathbb{N}$ be a multiplicatively large set and let*

$$F = \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}.$$

For any k polynomials $p_1(n), p_2(n), \ldots, p_k(n)$ which have positive leading coefficients and which satisfy $p_i(0) = 0$, $i = 1, 2, \ldots, k$, there exist $a, b \in \mathbb{N}$ such that $\{an_1^{p_1(b)}, an_2^{p_2(b)}, \ldots, an_k^{p_k(b)}\} \subset B$. Moreover, the set

$$\{b \in \mathbb{N} : \exists a \text{ with } \{an_1^{p_1(b)}, an_2^{p_2(b)}, \ldots, an_k^{p_k(b)}\} \subset B\}$$

is an additive IP set.*

Proof: The general scheme of the proof being identical to that of the proof of (ii) in Theorem 3.8, we shall formulate only the relevant recurrence theorem.

Let T_{n_i}, $i = 1, 2, \dots, k$ be the measure preserving transformations defined in the course of the proof of Theorem 3.8. It follows from Theorem 2.7 that for any $A \in \mathcal{B}$ with $\mu(A) > 0$, the set

$$\{b \in \mathbb{N} : \mu(A \cap T_{n_1}^{-p_1(b)} A \cap \dots \cap T_{n_k}^{-p_k(b)} A) > 0\}$$

is an additive IP* set. The rest of the proof is identical to the proof of statement (ii), Theorem 3.8. ∎

We will conclude this section by discussing some mixed "additive-multiplicative" configurations such as the AG sets defined in the introduction. The following theorem is an extension of Theorem 3.5 above.

THEOREM 3.10: *Let $\mathcal{S}^a$, $\mathcal{S}^m \subseteq \mathcal{F}$ be two families of finite subsets of $\mathbb{N}$ with the following properties:*

(i) *Any additively large set in $\mathbb{N}$ contains a configuration of the form $a + F$, where $F \in \mathcal{S}^a$.*

(ii) *Any multiplicatively large set in $\mathbb{N}$ contains a configuration of the form bF, where $F \in \mathcal{S}^m$.*

Then any multiplicatively large set E contains a configuration of the form $bF_2(a + F_1)$, where $F_1 \in \mathcal{S}^a$ and $F_2 \in \mathcal{S}^m$.

Proof: By using the Furstenberg correspondence principle and Theorem 3.1, we can find $F_1 \in \mathcal{S}^a$ such that for some $a \in \mathbb{N}$ the set $\bigcap_{n \in F_1} E/(a+n)$ is multiplicatively large. (See Remark 3.3.) By the assumption (ii), there exist $b \in \mathbb{N}$ and $F_2 \in \mathcal{S}^m$ such that $\bigcap_{n \in F_1} E/(a+n) \supset bF_2$. This implies that $E \supset bF_2(a + F_1)$ and we are done. ∎

Invoking Theorem 2.5, we get the following immediate corollary.

THEOREM 3.11: *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. Let $S_1, S_2 \subseteq \mathbb{N}$ be two infinite sets and let $IP^a(S_1)$ and $IP^m(S_2)$ be the additive and multiplicative IP sets generated by S_1 and S_2 respectively. Then for any $n \in \mathbb{N}$, there exist $a, b \in \mathbb{N}$, $d \in IP^a(S_1)$, and $q \in IP^m(S_2)$ such that*

$$\{bq^j(a + id), 0 \le i, j \le n\} \subset E.$$

It is known that for any finite partition $\mathbb{N} = \bigcup_{i=1}^r C_i$, one of the C_i is both additively and multiplicatively large and, in addition, contains additive and multiplicative IP sets (see Corollary 3.16 in [BH1] or Theorem 4.5 in [BH2]). Applying Theorem 3.11, we get the following application to partition Ramsey theory.

THEOREM 3.12: *Let $r, n \in \mathbb{N}$. For any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, \ldots, r\}$, $a, b \in \mathbb{N}$ and $d, q \in C_i$ such that*

$$\{bq^j(a + id), 0 \leq i, j \leq n\} \subset C_i.$$

The following result follows from Theorem 3.10 by invoking Theorem 2.6. Note that it contains Theorem 3.11 as a special case.

THEOREM 3.13: *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. For $k \in \mathbb{N}$, let $\sigma_\alpha^{(i)}$, $\alpha \in \mathcal{F}$, $i = 1, 2, \ldots, k$ be additive IP sets in $\mathbb{N}$, and let $\pi_\beta^{(j)}$, $\beta \in \mathcal{F}$, $j = 1, 2, \ldots, k$ be multiplicative IP sets. Then there exist $a, b \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{F}$ such that for all $i, j \in \{0, 1, \ldots, k\}$ one has $b\pi_\beta^{(j)}(a + \sigma_\alpha^{(i)}) \in E$.*

We shall conclude this Section with the proof of Theorem 1.5 from the introduction. In the course of the proof, we shall utilize the following multiplicative analogue of Theorem 3.1, the proof of which is practically identical with that of Theorem 1.1 in [B1] (our Theorem 3.1 above) and is omitted.

THEOREM 3.14: *Let $(X, \mathcal{B}, \mu)$ be a probability space and suppose that for some $c > 0$ and any $n \in \mathbb{N}$, the sets $A_n \in \mathcal{B}$ satisfy $\mu(A_n) \geq c$. For any Følner sequence (F_n) in $(\mathbb{N}, \cdot)$ there exists a set $P \subseteq \mathbb{N}$ such that $\overline{d}_{\{F_n\}}(P) \geq c$ and for any finite subset $F \subset P$, one has $\mu(\bigcap_{n \in F} A_n) > 0$.*

THEOREM 3.15: *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. For any $k \in \mathbb{N}$, there exist $a, b, d \in \mathbb{N}$ such that $\{b(a + id)^j, 0 \leq i, j \leq k\} \subset E$.*

Proof: Let $(X, \mathcal{B}, \mu, (T_n)_{n \in \mathbb{N}})$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ be the ergodic model which corresponds to the set E by Theorem 2.4. By the IP Szemerédi theorem, there exists a constant $c > 0$ such that the set

$$S = \{n : \mu(A \cap T_n^{-1}A \cap T_{n^2}^{-1}A \cap \cdots \cap T_{n^k}^{-1}A) > c\}$$

is a multiplicative IP* set and, hence, multiplicatively syndetic. Letting $A_n = A \cap T_n^{-1}A \cap T_{n^2}^{-1}A \cap \cdots \cap T_{n^k}^{-1}A$, $n \in S$, and applying Theorem 3.14 to the family $(A_n)_{n \in S}$, we get a set $P \subseteq S$ which, due to the syndeticity of the set S, is multiplicatively large and such that, for any finite $F \subset P$, $\mu(\bigcap_{n \in F} A_n) > 0$. We move now back to the set E. Letting $E_n = E \cap (E/n) \cap (E/n^2) \cap \cdots \cap (E/n^k)$ and invoking Theorem 2.4, we see that for any finite $F \subset P$, $\bigcap_{n \in F} E_n \neq \emptyset$. By Theorem 3.2, there exist $a, d \in \mathbb{N}$ such that $P \supset \{a + id, i = 1, 2, \ldots, k\}$. This implies that

$$S_k := \bigcap_{i=0}^{k} E_{a+id} = \bigcap_{0 \leq i,j \leq k} E/(a + id)^j \neq \emptyset.$$

Taking any $b \in S_k$ gives $\{b(a+id)^j, 0 \le i, j \le k\} \subset E$ and we are done. ∎

References

[B1] V. Bergelson, *Sets of recurrence of $\mathbb{Z}^m$-actions and properties of sets of differences in $\mathbb{Z}^m$*, Journal of the London Mathematical Society (2) **31** (1985), 295–304.

[B2] V. Bergelson, *Ergodic Ramsey Theory*, in *Ergodic Theory of $\mathbb{Z}^d$-actions*, London Mathematical Society Lecture Note Series 228, Cambridge University Press, Cambridge, 1996, pp. 1–61.

[B3] V. Bergelson, *Ergodic Theory and Diophantine problems*, in *Topics in Symbolic Dynamics and Applications*, London Mathematical Society Lecture Note Series 279, Cambridge University Press, Cambridge, 1996, pp. 167–205.

[B4] V. Bergelson, *Minimal idempotents and Ergodic Ramsey Theory*, in *Topics in Dynamics and Ergodic Theory*, London Mathematical Society Lecture Note Series 310, Cambridge University Press, Cambridge, 2003, pp. 8–39.

[BH1] V. Bergelson and N. Hindman, *A combinatorially large cell of a partition of* $\mathbb{N}$, Journal of Combinatorial Theory (Series A) **48** (1988), 39–52.

[BH2] V. Bergelson and N. Hindman, *Additive and multiplicative Ramsey theorems in* $\mathbb{N}$ *— some elementary results*, Combinatorics, Probability and Computing **2** (1993), 221–241.

[BHMP] V. Bergelson, B. Host, R. McCutcheon and F. Parreau, *Aspects of uniformity in recurrence*, Colloquium Mathematicum **84/85** (2000), part 2, 549–576.

[BL1] V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, Journal of the American Mathematical Society **9** (1996), 725–753.

[BL2] V. Bergelson and A. Leibman, *Set-polynomials and polynomial extension of the Hales–Jewett theorem*, Annals of Mathematics **150** (1999), 33–75.

[BL3] V. Bergelson and A. Leibman, *Topological multiple recurrence for polynomial configurations in nilpotent groups*, Advances in Mathematics **175** (2003), 271–296.

[BM1] V. Bergelson and R. McCutcheon, *Uniformity in the polynomial Szemerédi theorem*, in *Ergodic Theory of $\mathbb{Z}^d$-actions*, London Mathematical Society Lecture Note Series, 228, Cambridge University Press, Cambridge, 1996, pp. 273–296.

[BM2] V. Bergelson and R. McCutcheon, *Recurrence for semigroup actions and a non-commutative Schur theorem*, Contemporary Mathematics **65** (1998), 205–222.

[BM3] V. Bergelson and R. McCutcheon, *An ergodic IP polynomial Szemerédi theorem,* Memoirs of the American Mathematical Society **146** (2000), no. 695, viii+106 pp.

[F1] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions,* Journal d'Analyse Mathématique **31** (1977), 204–256.

[F2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory,* Princeton University Press, 1981.

[FK1] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations,* Journal d'Analyse Mathématique **34** (1978), 275–291.

[FK2] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory,* Journal d'Analyse Mathématique **45** (1985), 117–168.

[FK3] H. Furstenberg and Y. Katznelson, *A density version of the Hales–Jewett theorem,* Journal d'Analyse Mathématique **57** (1991), 64–119.

[FW1] H. Furstenberg and B. Weiss, *Topological dynamics and combinatorial number theory,* Journal d'Analyse Mathématique **34** (1978), 61–85.

[FW2] H. Furstenberg and B. Weiss, *Markov processes and Ramsey theory for trees,* Combinatorics, Probability and Computing (Ramsey Theory Special Issue) **12** (2003), 547–563.

[H] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, Journal of Combinatorial Theory (Series A) **17** (1974), 1–11.

[HiS] N. Hindman and D. Strauss, *Algebra in the Stone–Čech Compactification, Theory and Applications*, de Gruyter Expositions in Mathematics, 27, Walter de Gruyter & Co., Berlin, 1998. xiv+485 pp.

[K1] A. Y. Khintchine, *Three Pearls of Number Theory,* Graylock Press, Rochester, 1952.

[K2] A. Y. Khintchine, *Eine Verschärfung des Poincaréschen Wiederkehrsatzes,* Compositio Mathematica **1** (1934), 177–179.

[L] A. Leibman, *Multiple recurrence theorem for measure preserving actions of a nilpotent group,* Geometric and Functional Analysis **8** (1998), 853–931.

[S] E. Szemerédi, *On sets of integers containing no four elements in arithmetic progression,* Acta Mathematica Academiae Scientiarum Hungaricae **20** (1969), 89–104.

[W] B. van der Waerden, *Beweis einer Baudetschen Vermutung,* Nieuw Archief voor Wiskunde **15** (1927), 212–216.